SECTION 5.2: DEFINITE INTEGRALS

In this section we generalize the sums we used to determine areas. In the previous section, we assumed f to be a continuous function defined on a closed and bounded interval $[a, b]$ with the stipulation that $f(x) \geq 0$. Geometrically, this meant the graph of $y = f(x)$ never dipped below the x-axis.

Now consider what happens if the graph of $y = f(x)$ does dips below the x-axis. For the regions where $f(x) < 0$, we would be, in essence, **subtracting** that area away from the area above the x-axis, resulting in is termed the 'net' area between $y = f(x)$ and the x-axis.

Recall that in cases where we have uneven widths of rectangles, the notation '$\|\Delta\|$' denotes the width of the longest subinterval and is called the **norm** of the partition.

The notation '$\|\Delta\| \to 0$' is interpreted as saying 'the widest of the $\Delta x_i \to 0$'. If we insist the widest of the $\Delta x_i \to 0$, this necessarily forces the number of rectangles, $n \to \infty$. (Do you see why?)

We finally have the vocabulary to define the **Riemann Integral**:

$$\int_a^b f(x)\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

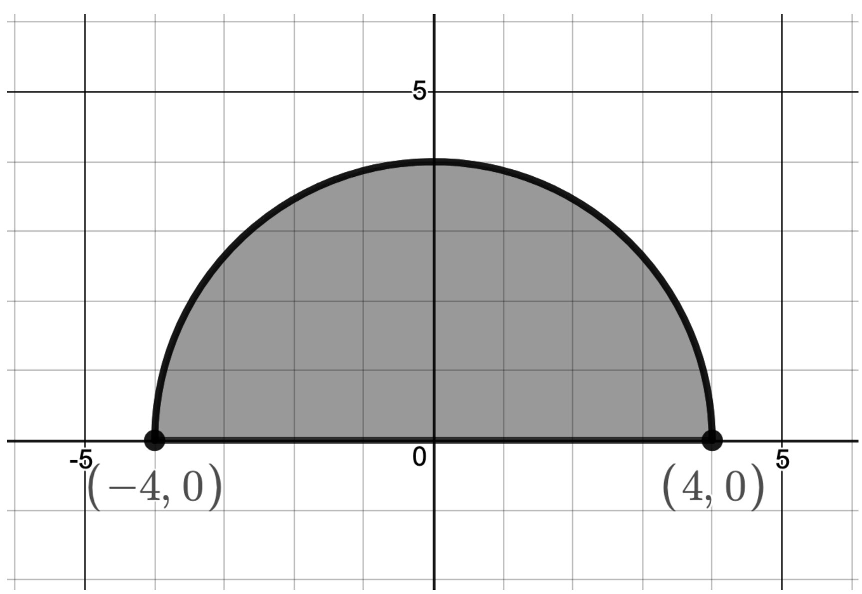
Geometrically:

$$\int_a^b f(x)\,dx = \text{net area between the graph of } f \text{ and the } x\text{-axis},$$

where, once again, 'net' area means the area between the graph of $y = f(x)$ and the x-axis which is above the x-axis minus the area between the graph of $y = f(x)$ and the x-axis which is below the x-axis.

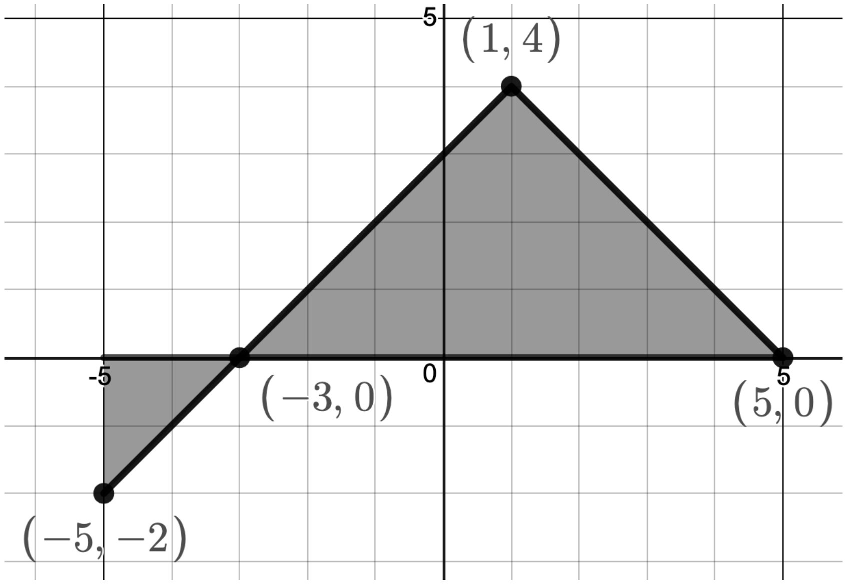
EXAMPLE 1: Evaluate the following definite integrals by interpreting each as a net area.

1. $\displaystyle\int_{-4}^{4} \sqrt{16 - x^2}\,dx$:



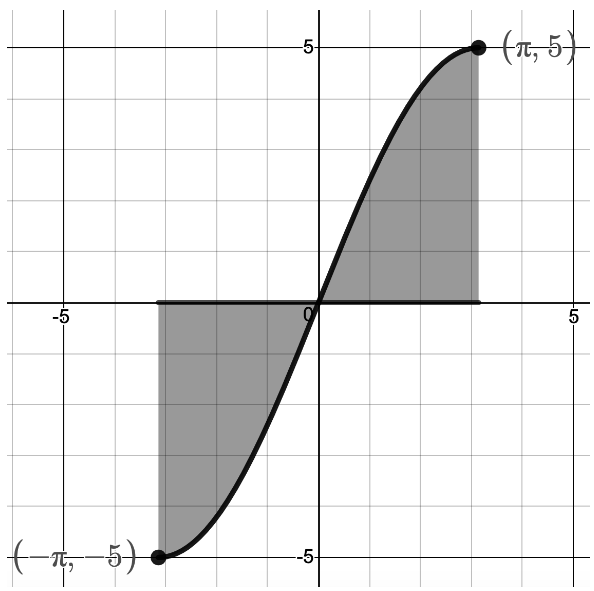
Ans: $\displaystyle\int_{-4}^{4} \sqrt{16 - x^2}\,dx = \frac{1}{2}\pi(4)^2 = 8\pi.$

2. $\int_{-5}^{5} (4 - |x-1|)\ dx$:



Ans: $\int_{-5}^{5} (4 - |x-1|)\ dx = -\frac{1}{2}(2)(2) + \frac{1}{2}(8)(4) = 14.$

3. $\int_{-\pi}^{\pi} \sin\left(\frac{\theta}{2}\right)\ d\theta$:



Ans: $\int_{-\pi}^{\pi} \sin\left(\frac{\theta}{2}\right)\ d\theta = 0$ since the areas cancel out (symmetry).

EXAMPLE 2: Find $\int_{-2}^{6} \frac{4-x}{2}\, dx$ by taking the limit of a right endpoint sum. Check your answer geometrically.

We first find a formula for RS_n and then take the limit as $n \to \infty$. We begin with

$$\Delta x_i = \frac{b-a}{n} = \frac{8}{n}, \quad \text{and} \quad x_i = a + i\Delta x_i = -2 + \frac{8i}{n}$$

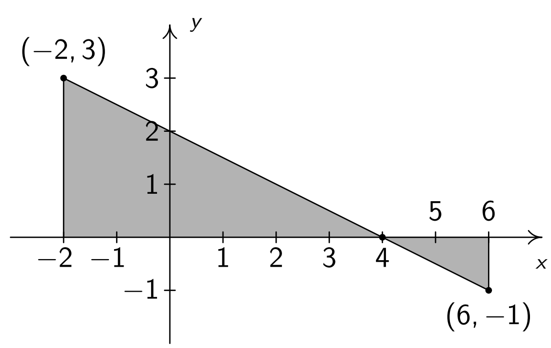
Next, $f(x_i) = \frac{4-x_i}{2} = \frac{1}{2}[4-x_i] = \frac{1}{2}\left[4 - \left(-2 + \frac{8i}{n}\right)\right] = 3 - \frac{4i}{n}$. Substituting gives:

$$\begin{aligned}
RS_n &= \sum_{i=1}^{n} f(x_i)\Delta x_i \\
&= \sum_{i=1}^{n} \left(3 - \frac{4i}{n}\right)\frac{8}{n} \\
&= \frac{8}{n}\sum_{i=1}^{n}\left(3 - \frac{4i}{n}\right) \\
&= \frac{8}{n}\sum_{i=1}^{n}(3) - \frac{8}{n}\sum_{i=1}^{n}\frac{4i}{n} \\
&= \left(\frac{8}{n}\right)(3n) - \frac{32}{n^2}\sum_{i=1}^{n} i \\
&= 24 - \left(\frac{32}{n^2}\right)\left(\frac{n(n+1)}{2}\right) \\
&= 24 - \frac{16(n+1)}{n}
\end{aligned}$$

We get $\lim_{n\to\infty} RS_n = 24 - 16 = 8$, so $\int_{-2}^{6} \frac{4-x}{2}\, dx = 8$.

To check our answer geometrically, we first graph $y = f(x) = \frac{4-x}{2}$ on $[-2, 6]$.

To find the 'net' area, we need to find the area between the graph and the x-axis which lies *above* the x-axis and subtract from that the area between the graph and the x-axis which lies *below* the x-axis. To do this, it is important to find the x-intercept of the graph (by setting $y = f(x) = 0$.) We get two triangles, as seen below.



The triangle which resides above the x-axis has area $\frac{1}{2}(6)(3) = 9$ square units. The triangle living below the x-axis has area $\frac{1}{2}(2)(1) = 1$ square unit. Since Net Area = Area Above the x-axis − Area Below the x-axis,

$$\int_{-2}^{6} \frac{4-x}{2}\, dx = 9 - 1 = 8.$$

EXAMPLE 3: (VIDEO) Find $\int_{-1}^{4} (2x-1)\,dx$ by taking the limit of a right endpoint sum.

Check your answer geometrically.

Ans: $\int_{-1}^{4} (2x-1)\,dx = 10$

Below we list several important properties of definite integrals. What do each of these mean geometrically?

PROPERTIES OF THE DEFINITE INTEGRAL

Suppose f and g are continuous functions. Then:

1. **Constant Multiple Rule:** $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$
2. **Sum and Difference Rule:** $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
3. **Area under a Point:** $\int_a^a f(x)\,dx = 0$
4. **Reverse Direction:** $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
5. **Additive Interval Property:** $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
6. **Definite Integrals and Symmetry:**
   - If f is **even**, that is if $f(-x) = f(x)$ for all x, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
   - If f is **odd**, that is if $f(-x) = -f(x)$ for all x, then $\int_{-a}^{a} f(x)\,dx = 0$.
7. **Preservation of Order:** Suppose f and g are continuous on $[a,b]$.
   - If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.
   - More generally, if If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

EXAMPLE 4: Suppose $\int_0^5 f(x)\,dx = 117$ and $\int_3^5 f(x)\,dx = -42$.

1. Find $\int_0^5 0.1\,f(x)\,dx$. Ans: $\int_0^5 0.1\,f(x)\,dx = 11.7$
2. Find $\int_0^3 f(x)\,dx$. Ans: $\int_0^5 f(x)\,dx = 159$
3. If f is even, find $\int_{-5}^{5} f(x)\,dx$. Ans: $\int_{-5}^{5} f(x)\,dx = 234$

HOMEWORK: Section 5.2: 37 - 65 odd, 75 - 81 odd